

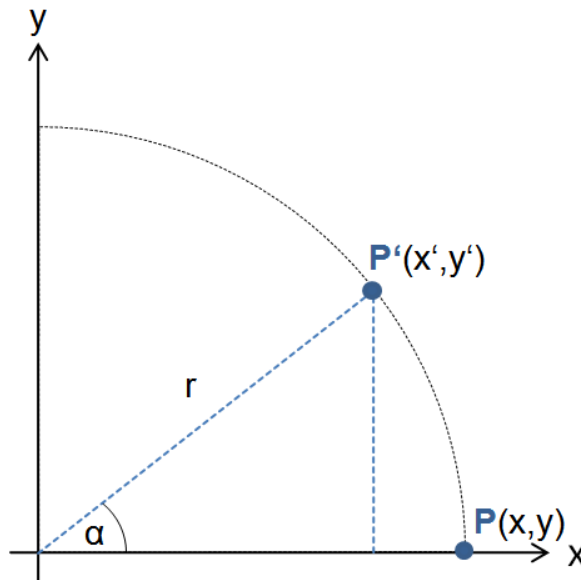
A short derivation to basic rotation around the x-, y- or z-axis

by Sunshine2k- September 2011

1. Introduction

This is just a short primer to rotation around a major axis, basically for me. While the matrices for translation and scaling are easy, the rotation matrix is not so obvious to understand where it comes from. And second, easy-to-understand derivations are rare and always welcome ☺? By just using basic math, we derive the 3D rotation in three steps: first we look at the two-dimensional rotation of a point which lies on the x-axis, second at the two-dimensional rotation of an arbitrary point and finally we conclude with the desired result of 3D rotation around a major axis.

2. 2D rotation of a point on the x-axis around the origin



The goal is to rotate point P around the origin with angle α . Because we have the special case that P lies on the x-axis we see that $x = r$.

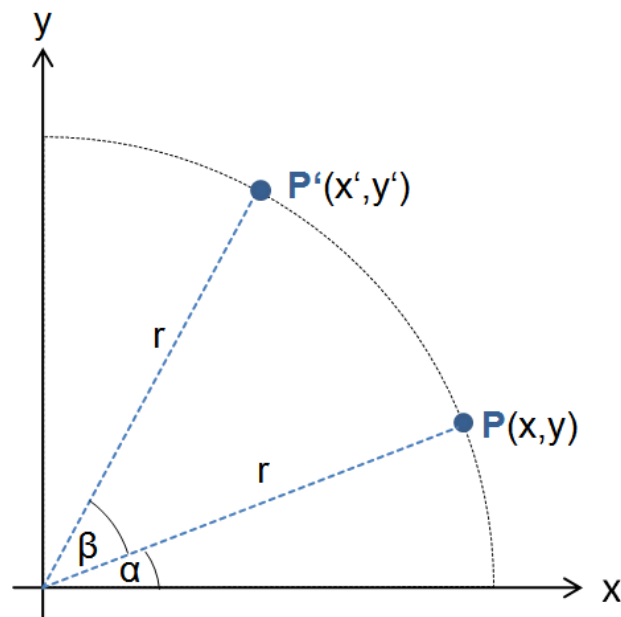
Using basic school trigonometry, we conclude following formula from the diagram.

$$\cos \alpha = \frac{x'}{r} \rightarrow x' = r * \cos \alpha = x * \cos \alpha$$

$$\sin \alpha = \frac{y'}{r} \rightarrow y' = r * \sin \alpha = x * \sin \alpha$$

That's it – we have a direct relation from the target point P' to P and angle α .

3. 2D rotation of an arbitrary point around the origin



This case is more general, the position of point P to rotate around the origin is arbitrary. What we can see directly from the diagram is following equations:

$$\begin{aligned}x &= r * \cos \alpha \\y &= r * \sin \alpha\end{aligned}$$

and

$$\begin{aligned}x' &= r * \cos(\alpha + \beta) \\y' &= r * \sin(\alpha + \beta)\end{aligned}$$

Now we need trigonometric identities (see [1]) to break down the equations:

$$\begin{aligned}x' &= r * \cos(\alpha + \beta) \\&= r * (\cos \alpha * \cos \beta - \sin \alpha * \sin \beta) \\&= r * \cos \alpha \cos \beta - r * \sin \alpha \sin \beta \\&= x * \cos \beta - y * \sin \beta\end{aligned}$$

Similarly, we do the corresponding transformation for y' :

$$\begin{aligned}y' &= r * \sin(\alpha + \beta) \\&= r * (\sin \alpha * \cos \beta + \cos \alpha * \sin \beta) \\&= r * \sin \alpha \cos \beta + r * \cos \alpha \sin \beta \\&= y * \cos \beta + x * \sin \beta\end{aligned}$$

This is exactly what we want because the desired point P' is described in terms of the original point P and the actual angle β . For conclusion in matrix notation:

$$\begin{bmatrix}x' & y'\end{bmatrix} = \begin{bmatrix}x & y\end{bmatrix} \begin{bmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{bmatrix}$$

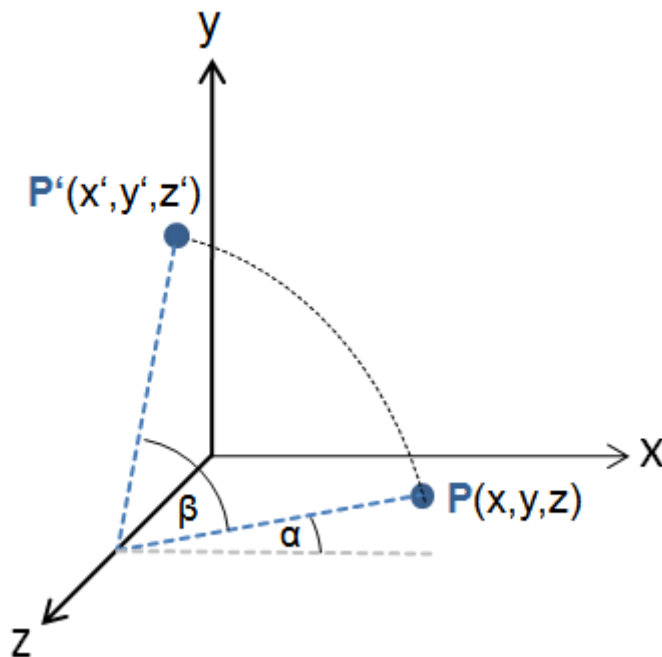
So what about the constraint that we just rotated around the origin? Well, in case of a rotation around an arbitrary point O' , just

1. Translate O' to the origin along with all other points.
2. Perform the rotation.
3. Inverse the translation to move O' and all other points to their initial positions.

4. 3D rotation around a major axis

So now you're excited at the fancy 3D rotation but in fact you know it already – because it's the similar as in the 2D case.

In 3D, the rotation is not defined by an angle and an origin point as in 2D, but by an angle and a rotation axis. If the rotation axis is restricted to one of the three major axis, then one component always remains same.



Look at the following (not optimal) figure where P is rotated around the z-axis: The z-component of the point remains same, so actually it's the same as rotating in the x-y-plane which corresponds to the 2D case.

Rotation around the z-axis in matrix notation (note $z' = z$):

$$[x' \quad y' \quad z'] = [x \quad y \quad z] \begin{bmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For completeness, here the rotation matrix around the x-axis

$$[x' \quad y' \quad z'] = [x \quad y \quad z] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{bmatrix}$$

And around the y-axis

$$[x' \quad y' \quad z'] = [x \quad y \quad z] \begin{bmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{bmatrix}$$

5. Summary

Hope you liked it ☺

Visit my homepage: <http://www.sunshine2k.de> or <http://www.bastian-molkenthin.de>

6. References

- [1] http://en.wikipedia.org/wiki/List_of_trigonometric_identities#Angle_sum_and_difference_identities