Notes about the (two-dimensional) perp dot product

Content

1. Motivation

This article presents my notes about the so-called perp dot product, or in other words the two-dimensional cross-product. Especially if you have asked yourself following questions, then you're exactly right here:

- What is the difference between the two-dimensional dot product, the two-dimensional perp dot product and the two-dimensional cross product?
- Based on the knowledge that the perp dot product of two 2D vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as $\mathrm{a_x} * \mathrm{b_y} - \mathrm{a_y} * \mathrm{b_x}$ **,** where does the connection to the sinus come from? In other words: Where does the definition $\mathrm{a_x} * \mathrm{b_y} - \mathrm{a_y} * \mathrm{b_x} = |\mathrm{a}| * |\mathrm{b}| * \sin \alpha$ come from?
- For what is the perp dot product useful?

I recommend to read my article "Derivation of the two-dimensional dot product" first
Note that vector are written as bold small letters, e.g. vector a is written by $\mathbf{a}$. The length of vector a is written as $|\boldsymbol{a}|$.

2. The perp dot product

The perp dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the dot product of the perpendicular vector of $\mathbf{a}$ and $\mathbf{b}$**.** Let's call the perpendicular vector of $\mathbf{a}$, the one that lies 90 degrees rotated counterclockwise to $\mathbf{a}$**,** vector $\mathbf{a}^{\perp}$ ("a perp").

Getting a perpendicular two-dimensional vector:

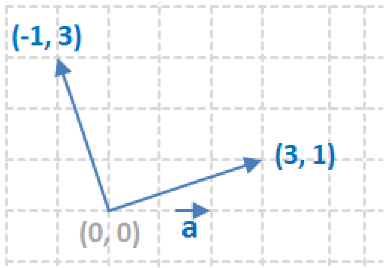
Given $\mathbf{a} = \binom{a_x}{a_y}$, then $\mathbf{a}^{\perp} = \binom{-a_y}{a_x}$, so simply exchange the x- and y-values and negate the first one.

This can be easily proved by using two-dimensional rotation matrices and rotating **a** by 90° counterclockwise:

$$(a_x \quad a_y) * \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = (a_x \quad a_y) * \begin{pmatrix} \cos 90° & \sin 90° \\ -\sin 90° & \cos 90° \end{pmatrix}$$

$$= (a_x \quad a_y) * \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (-a_y \quad a_x) = \mathbf{a}^{\perp}$$

Following example shows an example for $\mathbf{a} = \binom{3}{1}$ resulting in $\mathbf{a}^{\perp} = \binom{-1}{3}$:



Important Properties of $\mathbf{a}^{\perp}$*:*

- The length of a is equal to $\mathbf{a}^{\perp}$: $|\boldsymbol{a}| = |\mathbf{a}^{\perp}|$
- The perpendicular vector to $\mathbf{a}^{\perp}$ is $\mathbf{a}^{\perp\perp}$ which is – **a** (the opposite vector to a)
- Consequently, $\mathbf{a}^{\perp\perp\perp\perp}$ is again **a.**

Knowing what $\mathbf{a}^{\perp}$ means, we can calculate the dot product of two vectors $\mathbf{a}^{\perp}$ and **b**:
$\mathrm{perpDot}(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{\perp} \cdot \mathbf{b} = -a_y * b_x + a_x * b_y = a_x * b_y - a_y * b_x.$

This is called the perp dot product and it has some nice properties and applications as described in the following chapters.

3. The relation to the cross product

At first, recall the most important points of the cross product:

- The cross product of two vectors **a** and **b**, denoted as $\mathbf{a} \times \mathbf{b}$, is only defined in the three-dimensional space.
- The cross product is defined as:
  $$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} x \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$
- The result vector **c** of **a** x **b** is orthogonal (perpendicular) to the plane defined by **a** and **b** and the length of **c** is equal to the area of the parallelogram that the vectors **a** and **b** span.

So even the cross product is only defined in the three-dimensional space, let's try to apply it to the two-dimensional space by setting the "z-coordinate" of both vectors to zero, thus $a_z = 0$ and $b_z = 0$.

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_x \\ a_y \\ 0 \end{pmatrix} x \begin{pmatrix} b_x \\ b_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a_x b_y - a_y b_x \end{pmatrix}$$

My own interpretation that has nothing to do with a formal mathematical approach is as following: c_x and c_y are 0 because there is no vector **c** in the two-dimensional space that is orthogonal to **a** *and* **b** at the same time. It remains only the c_z value, so the result is not a vector anymore but a scalar value, $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ = $a_x b_y - a_y b_x$.

Conclusion: The "two-dimensional cross product" is equal to the two-dimensional perp dot product:

$$\mathbf{a} \times \mathbf{b} = \mathrm{perpDot}(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{\perp} \cdot \mathbf{b}$$

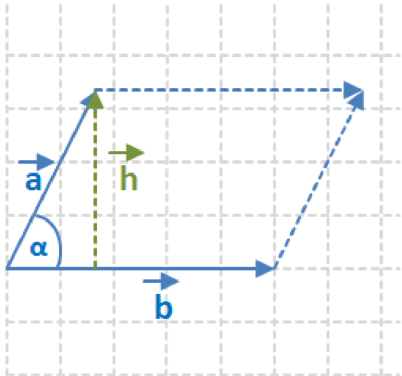
4. The relation to the angle between the vectors

Another property of the cross product is the following relation to the angle α between the vectors **a** and **b**:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| * |\mathbf{b}| * \sin\alpha$$

So the length of the cross product can be interpreted as the area of the parallelogram having vectors **a** and **b** as sides. This applies also for the "2D" case where the "z-coordinate" of both vectors are zero (like in the previous chapter).

Let's have a look at following figure:



The vectors **a** and **b** form a parallelogram.

The area of the parallelogram is the product of length of side b = width and height, thus area = $|\mathbf{b}| * |\mathbf{h}|$.

For angle **α** applies $\sin\alpha = |\mathbf{h}| / |\mathbf{a}|$, so Vector **h** can be defined as $|\mathbf{h}| = \sin\alpha * |\mathbf{a}|$.

Summarized, the area is $|\mathbf{a}| * |\mathbf{b}| * \sin\alpha$.

We know that in the two-dimensional space the cross product is the same as the perp dot product, we can also write:

$\mathrm{perpDot}$ (**a**, **b**) = $a_x * b_y - a_y * b_x = |\boldsymbol{a}| * |\boldsymbol{b}| * \sin\alpha$

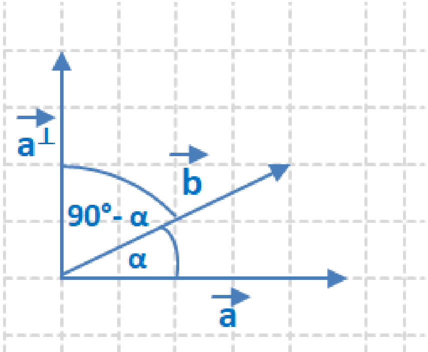
That is a direct relation between the perp dot product of two vectors and the angle between those vectors.

5. The relation of the angle in the dot product

The relation of the perp dot product to the dot product exists per definition:
$\mathrm{perpDot}(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{\perp} \cdot \mathbf{b}$. (see chapter 2)

We also seen in chapter 4 that
$\mathrm{perpDot}(\mathbf{a}, \mathbf{b}) = |\mathbf{a}| * |\mathbf{b}| * \sin\alpha$.

The dot product has a similar relation, but to the cosine of the angle between the two vectors:
$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| * |\mathbf{b}| * \cos\alpha$.
This can also be directly retrieved from the perp dot product as shown in following figure:



The angle between **a** and **b** is α. Then the angle between $\mathbf{a}^{\perp}$ and **b** is 90° - α because $\mathbf{a}^{\perp}$ is **a** rotated by 90°.

The dot product of **a** and **b** is
$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| * |\mathbf{b}| * \cos\alpha$.

Then dot product of $\mathbf{a}^{\perp}$ and **b** is
$\mathbf{a}^{\perp} \cdot \mathbf{b} = |\mathbf{a}| * |\mathbf{b}| * \cos(90° - \alpha)$.

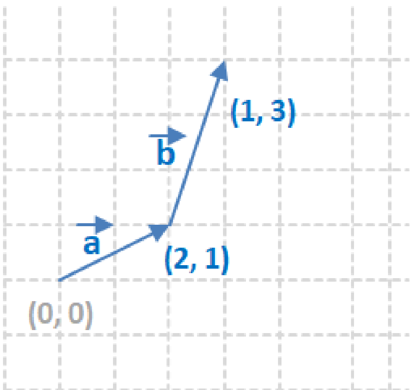
With the knowledge that $\cos\alpha \pm 90° = \sin\alpha$, it directly follows that
$\mathbf{a}^{\perp} \cdot \mathbf{b} = |\mathbf{a}| * |\mathbf{b}| * \cos(90° - \alpha) = |\mathbf{a}| * |\mathbf{b}| * \sin\alpha$

So the relation of the perp dot product to $\sin\alpha$ is a direct consequence of the definition of the dot product (and its relation to $\cos\alpha$) and the definition of $\mathbf{a}^{\perp}$.

6. Applications of the perp dot product

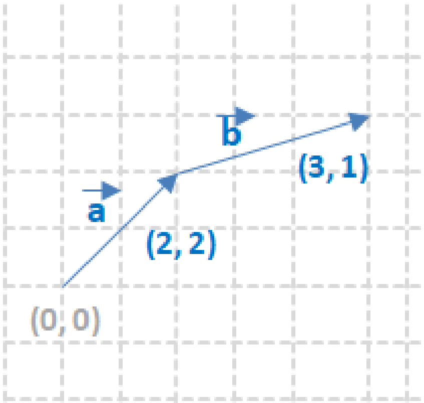
$$\boxed{\mathrm{perpDot}(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{\perp} \cdot \mathbf{b} = |\mathbf{a}| * |\mathbf{b}| * \sin\alpha}$$

6.1. “Direction turn” of two vectors



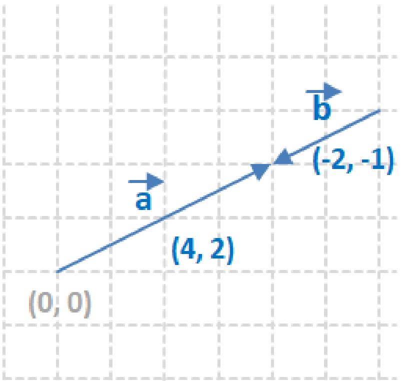
From the definition, it is concluded that the perp dot product is positive if **b** is counter clockwise to **a**. In other words, you move along vector **a** until the end where vector **b** begins - if you have to turn left to follow vector **b**, then the perp dot product is positive.

The case is shown in the figure on the left, where a “left turn” has to be done at the end of vector **a** to follow **b**. The perp dot product is $2 * 3 - 1 * 1 = 5$, thus positive.



The perp dot product is negative if **b** is clockwise to **a**. In other words, you move along vector **a** until the end where vector **b** begins - if you have to turn right to follow vector **b**, then the perp dot product is negative.

The case is shown in the figure on the left, where a "right turn" has to be done at the end of vector **a** to follow **b**.
The perp dot product is $2 * 1 - 2 * 3 = -4$, thus negative.



The perp dot product is zero if **a** and **b** have the same or opposite direction, thus are parallel .

In the figure on the left, the perp dot product is
$4 * (-1) - 2 * (-2) = 0$.

Summarized:

- $\mathbf{a}^{\perp} \cdot \mathbf{b} > 0$: **b** is counter clockwise from **a**.
- $\mathbf{a}^{\perp} \cdot \mathbf{b} < 0$: **b** is clockwise from **a**.
- $\mathbf{a}^{\perp} \cdot \mathbf{b} == \mathbf{0}$: **a** and **b** are parallel.

6.2. Polygon convexity

Above property about the direction of two vectors can also be used to find out if a polygon is convex. Move along all edges of the polygon, and for each two connected edges the perp dot product is calculated. If all products are greater zero or all products are less than zero, thus each perp dot product has the same sign, the polygon is convex.

6.3. Line intersection

The perp dot product is useful in the calculation of the intersection point of two lines or line segments in 2D. It can provide easily information if both lines are for example parallel or even coincident.

Check out my homepage for a detailed description of an algorithm to calculate the intersection point of two line segments.

7. Summary

This article provides the most important information about the perp dot product.

Hope you liked it!

Visit my homepage: http://www.sunshine2k.de or http://www.bastian-molkenthin.de.